In pursuit of the eulerianity of topological plane graphs

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Introduction

Given a topological plane graph $G = (V, E)$. The augmentation problem asks for a minimum object set to be added to G, such that the resulting graph attains a certain property. In particular, we look for the minimum cardinality edge set E', where $E' \cap E = \emptyset$, such that $G' = (V, E \cup E')$ is a topological eulerian (or semi-eulerian) plane graph.

The augmentation problem has been widely studied when the desired property is the k-connectivity of abstract, plane and geometric graphs. To the best of our knowledge, this is the first attempt to obtain the eulerianity of topological plane graphs by the addition of edges.

A graph G is eulerian, if it has a trail that starts and ends at the same vertex of the graph and uses exactly once every edge of G. The very well known characterization says that if all the vertices of G has even degree, then G is eulerian.

There are many straightforward examples of topological plane graphs where is clear that it is not feasible to obtain an eulerian plane graph. Such is the case of the odd degree plane stars, since the neighborhood of the central vertex of the star is completely saturated.

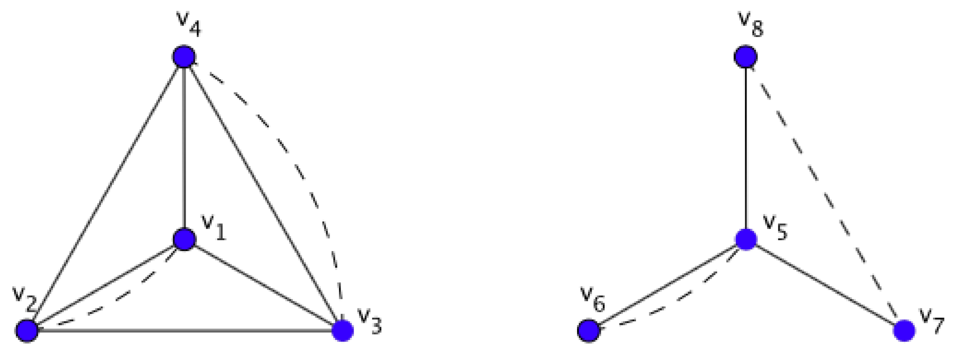

Figure 1: (Left) A plane embedding of k_4. (Right) An odd degree star.

We say that the complimentary graph $\overline{G}$ of $G = (V, E)$, is the graph having the vertex set V, and the edge set $\overline{E} = \{e \notin E \mid G \cup e \text{ is plane}\}$. We restrict the augmentation problem to those graphs where the number of odd vertices in every connected component of its complimentary graph is even.

First, we show that the plane eulerianity augmentation problem on topological graphs is feasible, when the input graph is a plane tree. Second, we prove that the plane eularianity augmentation problem is *NP-Complete*, when the input is an arbitrary topological graph. Third, we give a lower bound on the number of edges necessary to be added to attain a topological eulerian plane graph. Finally, we shown that the addition of edges it is not always sufficient to achieve the eulerianity of topological plane graphs, even if we look for the semi-eulerianity instead of eulerianity.

Plane eulerianity augmentation in topological trees

Theorem 1: *Let $T = (V, E)$ be a topological plane tree, and $|V| = n$. Then, T can be augmented to a topological eulerian plane graph G' with the addition of $\frac{k}{2}$ edges, where $k \leq n$ is the number of odd degree vertices in T.*

sketch of proof: Let V_o be the set of odd degree vertices in T, $|V_o| = k$, and F_T the unique face of T. Note that any two vertices can be join by a curve edge, since all of them are incident to the same face. We join two vertices v_i,v_j whose distance is at least three and $T \cup v_iv_j$ has all of its vertices incident to the face F_T.

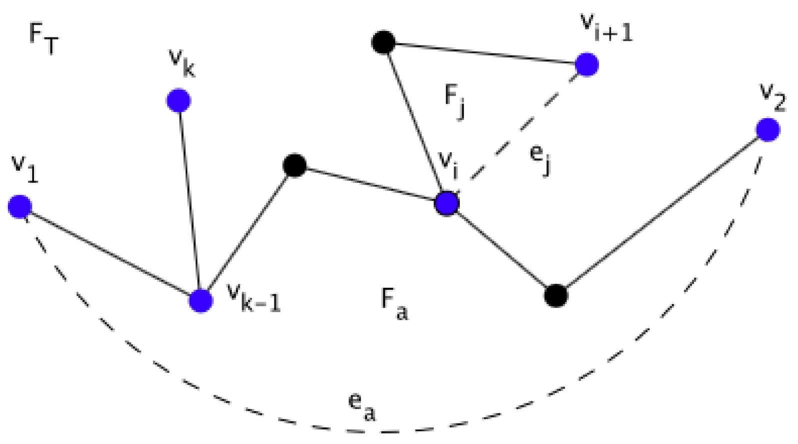

Figure 2: A plane topological tree. The odd degree vertices are marked as blue disks. The dashed edges represents objects added by the augmentation process.

We continue joining the k-2 vertices as follows. We join two vertices v_i', v_j' which distance is at least two and $T \cup v_i'v_j'$ has all of its vertices incident to the face F_T. If the last couple of vertices does not meet the first condition then we use the first couple of joined vertices. □

NP-Completeness of the plane eulerianity augmentation in topological graphs

Theorem 2: *Let $G = (V, E)$ be a topological plane graph. The problem of deciding if G can be augmented to a topological eulerian plane graph G' is NP-Complete.*

sketch of proof: We reduce *3SAT* to the plane eulerianity augmentation problem in topological graphs. Given a *3SAT* formula Φ, we construct a graphs in the following manner. We define two subgraphs or gadgets, the first to represent a variable (configured according to its corresponding value) and the second to represent a clause.

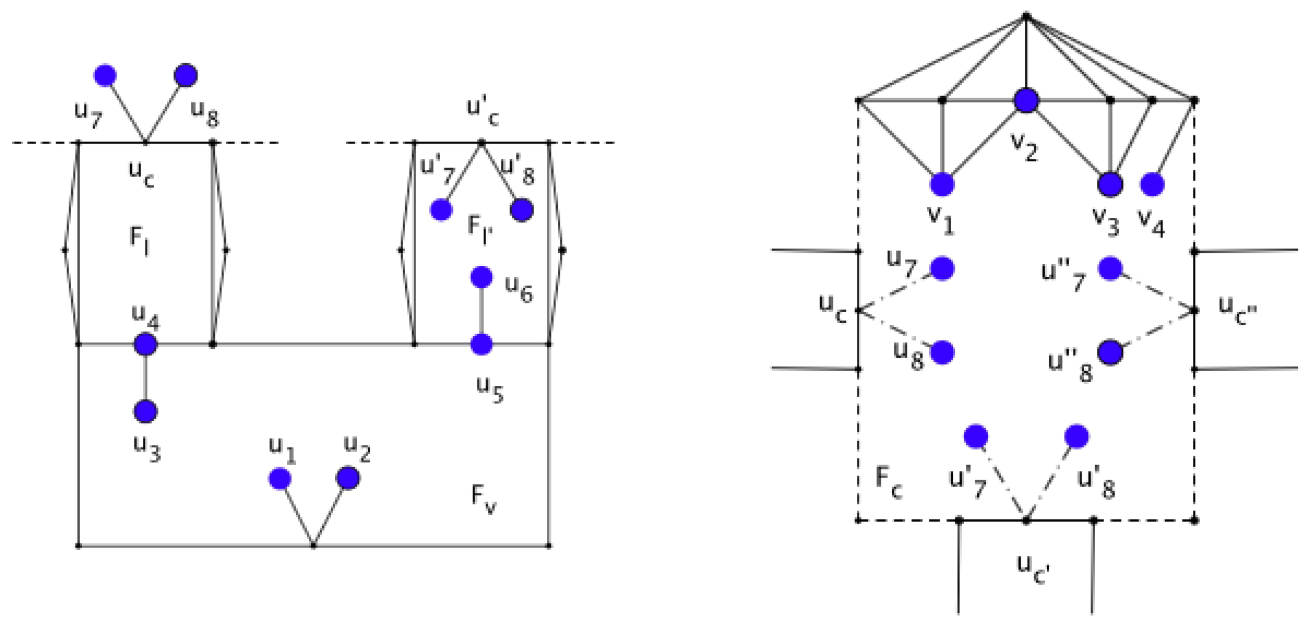

Figure 3: (Left) Variable gadget set to true. (Right) Clause gadget.

The variable gadget has three leaf vertices inse the left tower (resp. right tower) that act as true/false switches. A variable gadget is set to true when its left tower embeds the vertices u_7,u_8 into the face of the corresponding clause gadget. The clause gadget is configured in such way that it is not posible, by its own, to connect the vertices v_1, v_2, v_3 and v_4 with two edges avoiding edge crossings. Therefore, it is necessary at least one variable gadget transmitting the value true. The proof follows since if it is possible to find a setting of the variables such that Φ is satisfied, then G can be augmented to a topological eulerian plane graph with the minimum set of edges. □

Lower bound of the plane eulerianity augmentation problem

Theorem 3: *There exists topological plane graphs whose augmentation to a topological eulerian plane graph requires $\frac{11n}{15}$ edges.*

sketch of proof: Our basic subgraph is builded by 15 vertices, 12 of them are odd degree vertices. Every leaf is embedded independently into a triangular face. The 3 most inner odd degree vertices are placed such that only a couple can be join directly by an edge, same as the 3 most outer odd degree vertices.

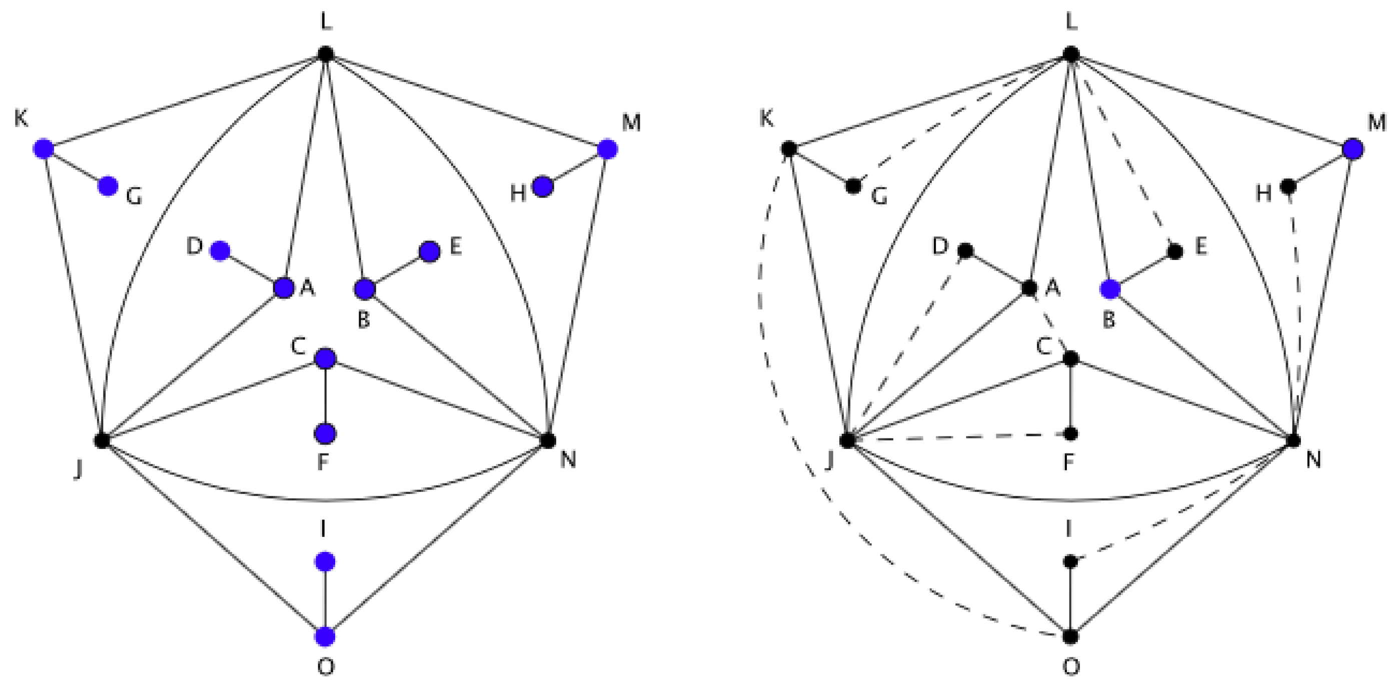

Figure 4: (Left) A graph that requieres 12 edges to be augmented. (Right) Partial augmentation. The remaining two odd degree vertices need to be join by one of the following paths (B, C), (C, L), (L, O), (O, M) or (B, A), (A, N), (N, K), (K, M).

Note that any two leaves can be join by a path of two edges. We can also join two outer/inner vertices by an edge. Finally, the remaining two vertices cannot be join by a path smaller than 3 edges. We claim that our basic graph can not be augmented to an topological eulerian plane graph with less than 12 additional edges. To build an arbitrary large graph we take an even triangulation (i.e. all of its vertices having even degree) of n vertices and we replace every vertex by our basic subgraph. □

Finally we show that there exists graphs that cannot be augmented to a topological eulerian plane graph, even if we soft the property and we look for topological semi-eulerian plane graphs.

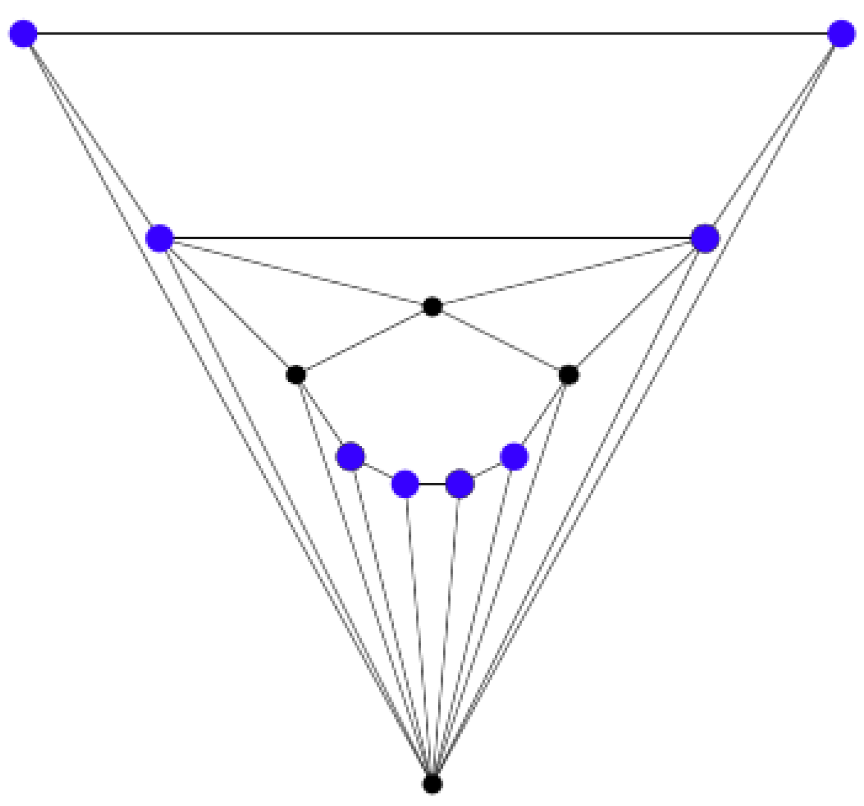

Figure 5: Graph that cannot be augmented to a topological eulerian plane graph.

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