Planarity Preserving Augmentation of Topological and Geometric Plane Graphs to Meet Parity Constraints

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Abstract

We introduce the augmentation problem to meet parity constraints in topological and plane geometric graphs. We show a family of plane topological graphs such that any augmentation leaves at least $\frac{2n}{5}$ vertices without meeting their parity constraints, and a family of plane geometric trees such that any augmentation leaves at least $\lfloor \frac{n}{10} \rfloor$ vertices without meeting their parity constraints. We prove that the problem of adding a minimum number of edges to plane topological graphs is $\mathcal{NP}$-Hard. When the input graph is a topological tree finding a minimum set of edges that needed to be added to meet a parity constraint is solvable in $\mathcal{O}(n)$ time and $\mathcal{O}(1)$ space. We also establish a lower bound of $\lceil \frac{11n}{15} \rceil$ on the number of necessary edges to augment a topological graph when the graph is augmentable, and a lower bound of $\lceil \frac{6n}{11} \rceil$ on the number of necessary edges to augment a geometric tree when the tree is also augmentable to meet the parity constraints.

1 Introduction

A *topological graph* is a graph together with an embedding on the plane, such that the vertices are represented by distinct points and the edges are represented by Jordan arcs connecting pairs of vertices.

A *geometric graph* is a graph in which its vertices are represented by points on the plane, and its edges by straight line segments joining pairs of vertices. A *planar* graph is a graph that can be embedded in the plane in such a way that its edges may intersect only at their endpoints. Such an embedding is called a *planar embedding* of the graph.

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A *plane graph* is a planar emmbeding of a planar graph, and we refer to its points as vertices and lines as edges. Two or more geometric graphs are *compatible* if their union is a plane geometric graph.

Given a plane topological (resp. geometric) graph $G = (V, E)$ and a set of parity constraints $C = \{c_1, c_2, ..., c_n\}$ where each $v_i \in V$ has assigned the constraint c_i (to be of degree odd or to be of degree even), the *augmentation problem to meet parity constraints* is that of finding a set of edges E', where $E' \cap E = \emptyset$, such that:

1. $G' = (V, E \cup E')$ is a plane topological (resp. geometric) simple graph.
2. The degree of each vertex $v_i \in G'$ meets its parity constraint c_i.

Observe that if a vertex of G does not meet its parity constraint, then its degree must increase by an odd integer. In what follows we will denote by P the set of vertices of G that do not satisfy its degree constraints in $C = \{c_1, c_2, ..., c_n\}$. Let H be the graph with vertex set V, and edge set E'. The degree of each vertex in H is odd, and thus H has an even number of vertices.

We say that the neighborhood of a vertex is *saturated* if there is no edge that can be added to G, incident to v, and avoiding edge crossings. For example, if G is a planar graph and the subgraph induced by v_i and its neighbors is a wheel with no other vertices inside it, then the neighborhood of v_i is saturated, or for short v_i is saturated. Thus from now on we will assume that the degree of any vertex in P is smaller than $n - 1$ and its neighborhood is not saturated.

It is easy to see that there are many planar graphs that cannot be extended to meet a set of parity constraints. For example take a planar graph that is a triangulation Δ minus two edges $e = (u, v)$ and $e' = (x, y)$ such that u, v, x, and y are different vertices. Then we cannot change the parities of u and x without breaking the planarity of Δ.

Then, the graphs we study in this paper must have the following properties:

1. The graphs are simple.

2. If a vertex is in P, its degree is smaller than $n-1$ and its neighborhood is not saturated.

Let $G = (V, E)$ be a topological (resp. geometric) simple plane graph, and P the set of vertices not meeting their parity constraints in G. The complementary graph $\overline{G} = (V, \overline{E})$ of G, is composed by the set of vertices V and all the edges $e \notin E$ that can be added to G in such a way that $G \cup e$ is plane.

In all the figures throughout this paper, the vertices in P are represented as empty discs, and the dashed edges represent edges added by the augmentation process.

2 Planarity Preserving Augmentation of Topological Graphs

First we consider the case when the input graph is a plane topological tree.

Theorem 1 *Let $T = (V, E)$ be a plane topological tree that is not a star, and that we want to augment to a plane graph with a set of parity constraints $C = \{c_1, c_2, ..., c_n\}$. Then T can always be augmented to meet its parity constraints in $\mathcal{O}(n)$ time, with the addition of at most $\frac{k}{2} + 1$ edges, where $|P| = k$.*

Proof. Let $T' = (V', E')$ be the minimal topological connected subgraph of T containing all the elements in P. If T' is a star, our problem can be solved easily by using a vertex v in T not adjacent to the center of the star, see Figure 1. Suppose then that T' is not a star.

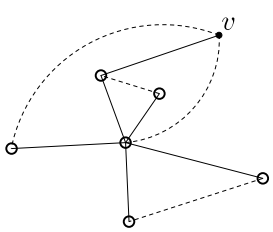


Figure 1: Augmentation of T' when it is a star.

Let $F_{T'}$ be the unique face in T'. It is easy to see that the nodes in T not in T' can be discarded now, and reinserted again once we finish our augmentation process in T'. Note first that any pair of vertices in P can be joined by a Jordan curve contained in $F_{T'}$.

Let $v_1, v_2 \in P$ be two vertices such that their edge distance in T' is at least three, and such that when we add the edge (v_1, v_2) to T' we form a cycle $\mathcal{C}$ containing no vertex of T' in its interior. Such vertices exist, for otherwise, T' would be a star.

Let F_a be the face bounded by $\mathcal{C}$. See Figure 2.

The addition of e_a changes the parities of v_1 and v_2, therefore these two vertices no longer belong to P.

We proceed by iteratively looking for a pair of vertices $v_i, v_{i+1} \in P$, such that their edge distance in T' is at least two and such that the cycle obtained by adding the edge $e_j = (v_i, v_{i+1})$ contains no vertices of P in its interior. These vertices will exist as long as we have at least four vertices in P whose parities have not changed.

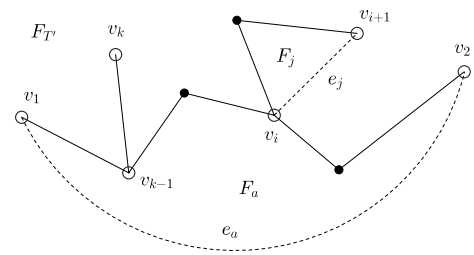


Figure 2: A topological tree.

Let v_{k-1}, v_k be the last two vertices in P. Note that these two vertices form the only one pair that could not be at distance two. Then, we use the first pair of joined vertices v_1, v_2 as follows. Since v_1 is at least at distance three from v_2 then we have the following cases:

- *Case 1:* v_1 is at least at distance two from v_{k-1} and v_2 is at least at distance two from v_k. Then, we can exchange e_a by (v_1,v_{k-1}) and (v_2,v_k), or by (v_1,v_k) and (v_2,v_{k-1}).
- *Case 2:* Suppose w.l.o.g. that v_1 is at distance one from v_{k-1} and v_2 is at least at distance two from v_k. Then, we can exchange e_a by (v_1,v_k) and (v_2,v_{k-1}).
- *Case 3:* Suppose w.l.o.g. that v_1 is at distance one from v_{k-1} and v_2 is at distance one from v_k. Then, we can exchange e_a by (v_1,v_k) and (v_2,v_{k-1}).

It is not hard to see that the above process can be carried out in linear time.

It follows that by adding a compatible matching of the vertices in P, or a compatible matching that covers all the vertices of P but two (which can be joined by a path with two edges) we have a topological graph obtained by the addition of at most $\frac{k}{2}+1$ edges. Moreover the bound is tight since there is no way to augment a topological tree with k vertices in P with less than $\frac{k}{2}$ edges. □

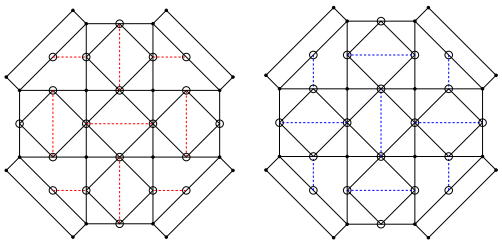
Next, we present a result about the hardness of the augmentation problem to meet parity constraints.

Theorem 2 *Let $G = (V, E)$ be a plane topological graph and C a set of parity constraints assigned to V. The problem of deciding if there exists a set E' with the minimum number of edges such that, $G' = (V, E \cup E')$ is a plane topological graph meeting all the parity constraints in C while preserving its embedding is $\mathcal{NP}$-Hard.*

Proof. We do the proof by reducing the Planar 3SAT Problem to the augmentation problem to meet parity

constraints. Given a planar 3SAT formula Φ, we build a plane topological graph G_Φ that we want to augment with the minimum number of edges such that all of its vertices meet their parity constraints.

We define three subgraphs, or gadgets, as building blocks for our reduction: The basic gadget, the literal gadget and the clause gadget. The basic gadget consists of a graph which has only two possible augmentations: Positive (in red) and Negative (in blue), as illustrated in Figure 5.

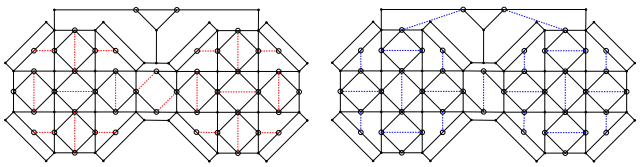


(a) (b)

Figure 3: Possible augmentations of the basic gadget: (a) Positive. (b) Negative.

A literal gadget is a subgraph that has butterfly shape, see Figure 4. The wings are two basic gadgets, the body has two white vertices, and the antennae has other two white vertices. Both wings must have the same augmentation, either positive or negative. Accordingly, we call the former a positive augmentation and the latter a negative augmentation of the literal gadget.

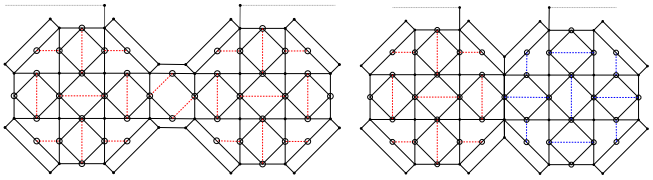
Note that when the literal is assigned a positive value, the antennae of the butterfly remain without changing their parity.



(a) (b)

Figure 4: Possible augmentations of a literal gadget. (a) A positive augmentation. (b) A negative augmentation

For each occurrence of a variable x in Φ we will have a literal gadget. All the literal gadgets of the same variable will be joined to form a chain of butterflies, where the right wing of a butterfly will be joined to the left wing of the next butterfly. Two consecutive butterflies will be joined as follows: If in both occurrences x is negated or non-negated then they will be joined as illustrated in Figure 5a. Otherwise, they will be joined as illustrated in Figure 5b. Finally, we join the leftmost



(a) (b)

Figure 5: The union of two literals of the same variable: (a) When both occurrences are negated or non-negated (with a positive augmentation). (b) When one occurrence is negated and the other is not (with a positive augmentation, left wing and negative augmentation, right wing).

wing with the rightmost wing with a band. If the obtained chain of butterflies has an odd number of white vertices we add an extra white vertex inside the band as illustrated in Figure 6.

A clause gadget F_c is joined with three literal gadgets ℓ_i, ℓ_j, and ℓ_k, an example of a clause is illustrated in Figure 6. We say that F_c has true value if the white vertices of F_c are matched with two vertices of the literal gadgets having positive value, otherwise F_c has false value. For example, if ℓ_i is the only one having positive value, then you can join the two vertices of the antennae of ℓ_i with the two white vertices of F_c changing their parity and F_c has true value. If all the literals have negative value, then the two white vertices of F_c cannot be augmented with only one edge and F_c has false value. It is straightforward to see that all the other cases can be solved leaving F_c with true value.

The proof follows since if it can be found a set with the minimum number of edges to augment G_Φ to meet its parity constraints, then, an assignation of values that satisfies Φ is obtained. □

Next, we present a family of plane topological graphs to establish a lower bound on the number of edges needed to be added to a plane topological graph to meet a set of parity constraints, while preserving their planarity.

Theorem 3 *There exists a family of plane topological graphs G with n vertices such that any augmentation of them in which all of its vertices have even degree requires the addition of at least $\lceil \frac{11n}{15} \rceil$ edges.*

Proof. Consider the graph shown in Figure 7. Such graph has 15 vertices, 12 of which are odd degree vertices. We claim that this graph cannot be augmented to meet its parity constraints with less than 12 edges.

Each leaf of the graph is enclosed in a triangular face. Note that there is no way to join a leaf with any other odd degree vertex using only one edge avoiding crossings. Note also that there is no way to join two inner

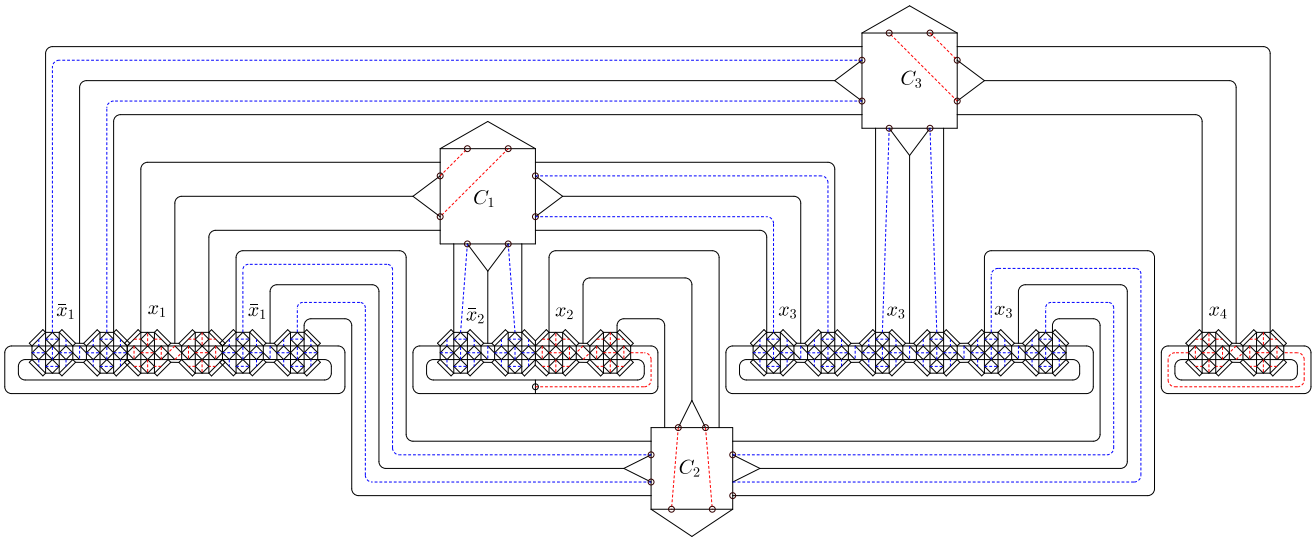


Figure 6: Graph G_Φ for $(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4)$, augmented in accordance to the assignment $(x_1, x_2, x_3, x_4) = (T, T, F, T)$. The distribution of the clause and literal gadgets is consistent with the topology of the given 3SAT plane graph for Φ.

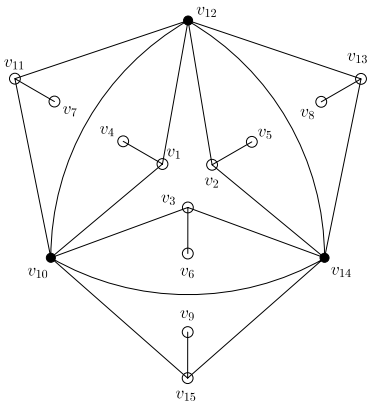


Figure 7: A plane topological graph G that requires at least 12 edges to augment such that all of its vertices end with even degree.

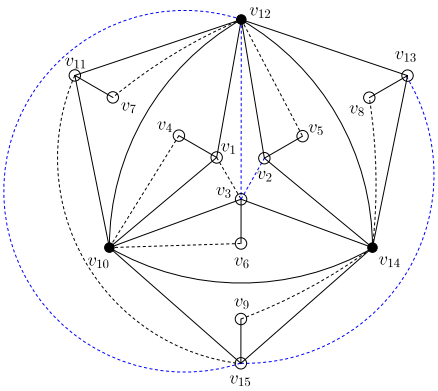


Figure 8: The dashed lines change the parities of all the odd vertices of G.

odd degree vertices with two outer odd degree vertices with disjoint paths. Therefore, two inner odd degree vertices have to be joined with an edge. The same happens for the three outer odd degree vertices. Figure 8 shows an example of the previous described augmentation. Finally, there are two odd degree vertices, one in the inner face and one in the exterior face of the graph, such that the only way to change their parity is to join them with a path with 4 edges, depicted in Figure 8 with dashed blue lines. Therefore we require a total of 12 edges to augment the graph in order to meet its parity constraints.

Now consider an even triangulation with n vertices (a triangulation is said to be even if all its vertices have even degree). Assign to each vertex of the triangulation a copy of the graph shown in Figure 7. Embed each copy in one of the adjacent faces of its assigned vertex, in such a way that there are no two copies inside of the same face, we can do this because the triangulation has $2n-4$ faces. Replace each vertex of the triangulation by the vertex v_{10}, v_{12} or v_{14} of its assigned copy. Figure 9 shows an example of this construction.

Note that the augmentation of the leaves and the most inner odd degree vertices of each copy (save the vertices that correspond to v_2 and v_{13}) requires the same number of edges as in Figure 8. We can join a copy of the vertex v_2 to a copy of the vertex v_{13} with a path of length 3 instead of a path of length 4, as shown in Figure 9, thus saving one edge per pair. It follows that it is required $\lceil \frac{11n}{15} \rceil$ edges to augment such graph to meet its parity constraints. □

Theorem 4 *There is a family of topological graphs such that it is not possible augment it to change the parity of all of its n vertices. In these graphs, we can change the parity of at most $\frac{2n}{5}$ of its vertices.*

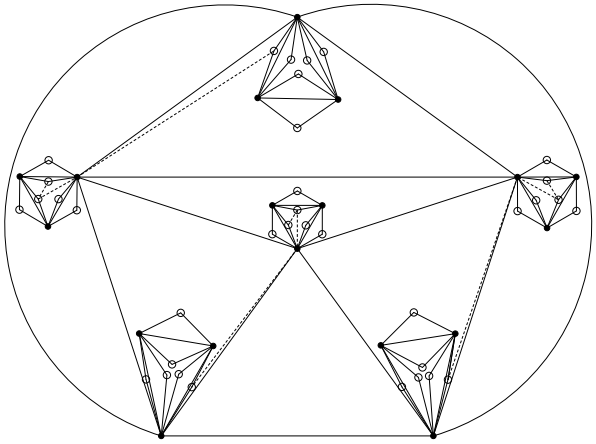


Figure 9: A plane topological graph requiring $\lceil \frac{11n}{15} \rceil$ edges to augment it to an Eulerian plane topological graph.

Proof. Our family of graphs is constructed as follows: Take k disjoint pentagons, and construct a graph whose vertices are the vertices of our pentagons. Add enough edges until the exterior of the union of the pentagons is triangulated, see Figure 10.

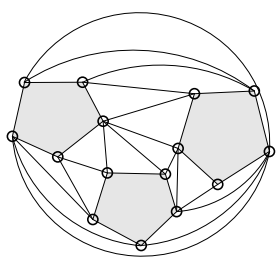


Figure 10: A plane topological graph in which any augmentation leaves $\frac{2n}{5}$ vertices without meeting its parity constraints.

It is easy to see that we can change the parity of only two of the vertices of each pentagon, that is, only $\frac{2n}{5}$ of the vertices of graph. □

3 Planarity Preserving Augmentation of Geometric Graphs

First we consider the special case when the input graph is a plane geometric tree. We show that the family of geometric trees proposed by C. Toth in [1], and refined later by A. García and J. Tejel in [2], establishes a lower bound on the number of edges needed to augment a plane geometric tree to meet a set of parity constraints while preserving its planarity.

Theorem 5 *There exists a family of plane geometric trees such that to augment them to meet a set of parity constraints requires the addition of* $\lceil \frac{6n}{11} \rceil$ *edges.*

Proof. In the family of trees that we will generate, we want to change the parity of all of its vertices of odd degree.

The basis of our construction is a tree similar to that introduced in [2]. It consists of a tree with 7 leaves and 8 vertices of odd degree shown in Figure 11.

Since the four internal leaves of the tree are placed in such a way that they cannot see each other, the only two edges joining two odd degree most inner vertices are (h_2, v_1) and (h_3, v_1). In the exterior, only two of the three external leaves, h_5, h_6, and h_7, of the tree can be joined by an edge. Thus, we can add one edge in the interior of the tree, and one edge between two external leaves joining odd degree vertices. Suppose w.l.o.g. that we joined h_3 to v_1 and h_6 to h_7.

Since the 4 odd degree vertices remaining do not have direct visibility with each other, then at least two edges to join each pair of them are needed. Therefore, to augment the basic construction, 6 edges are necessary.

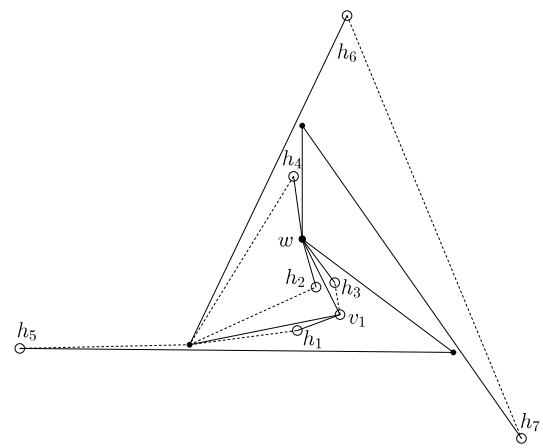


Figure 11: A plane geometric tree that requires six additional edges to augment it to a graph in which all of its vertices have even degree.

To generalize the construction to a family of plane geometric trees, we take a copy scaled down of the basic construction and we embed it attached to vertex w as shown in Figure 11). Note that vertex w becomes an odd degree vertex.

If we iterate this process, at each iteration we add 11 vertices, and create 8 vertices with odd degree. It follows that to augment any member of the obtained family of trees in which all of their vertices have even degree we need at least $\lceil \frac{6n}{11} \rceil$ edges. □

We show now a family of geometric trees in which not all of their vertices can change parity when we add edges to them. Such a family was initially proposed by García, Huemer, Hurtado and Tejel in [3].

Theorem 6 *There exists a family of plane geometric trees, such that no matter how we augment them, at*

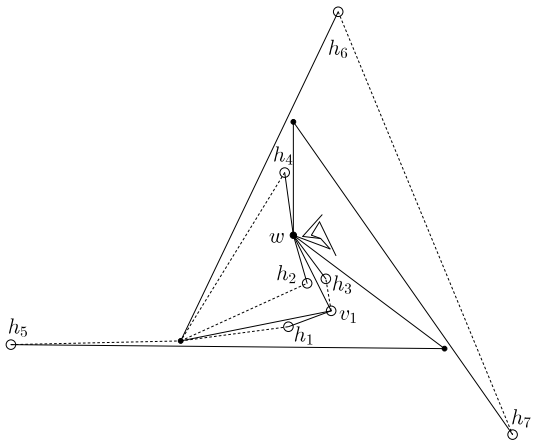


Figure 12: Constructing a family of plane geometric trees that require $\lceil \frac{6n}{11} \rceil$ additional edges in order to meet their parity constraints.

least $\lfloor \frac{n-1}{10} \rfloor$ *of its vertices are left with their parity unchanged.*

Proof. Consider the geometric tree T shown in Figure 13. We want to show that no matter how we augment it, at least $\lfloor \frac{n}{10} \rfloor$ of its vertices remain with odd degree. Note that in the complementary graph $\overline{G}$ of T, v_1 has degree 1 as it only sees vertex z (apart from its neighbors in T). A symmetric situation happens with v_2. If (v_1,z) is selected to augment T, then (v_2,w) cannot be part of such augmentation since both edges cross each other. Thus in any augmentation of T one of v_1 or v_2 remains unchanged.

We can replicate the previous situation to build an arbitrarily large tree as shown in Figure 14. In this manner, any augmentation leaves at least one odd vertex per pair of degree-3 vertices.

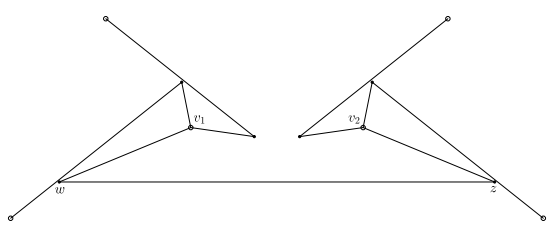


Figure 13: In any augmentation of this graph, at least one of v_1 and v_2 cannot change its parity.

It follows that any augmentation of T leaves at least $\lfloor \frac{n-1}{10} \rfloor$ odd degree vertices without meeting their parity constraints. □

4 Conclusions

In this paper, we studied the augmentation problem to meet parity constraints in topological and plane geometric graphs. We obtained a family of plane topological graphs with a set of parity constraints such that any augmentation of them leaves at least $\frac{2n}{5}$ vertices without meeting their parity constraints. We also obtained a family of plane geometric trees such that any augmentation leaves at least $\lfloor \frac{n-1}{10} \rfloor$ vertices without meeting their parity constraints. We also proved that the complexity of finding the smallest number of edges needed to augment a plane topological graph to meet a set of parity constrains is $\mathcal{NP}$-Hard.

The case in which the input graph is a topological tree, the problem is always solvable with the minimum number of additional edges in $\mathcal{O}(n)$ time and $\mathcal{O}(1)$ space. We also established a lower bound of $\lceil \frac{11n}{15} \rceil$ on the number of necessary edges to augment a topological graph when the graph is augmentable, and a lower bound of $\lceil \frac{6n}{11} \rceil$ on the number of necessary edges to augment a geometric tree when the tree is also augmentable. Finding upper bounds in the two previous problems is still open.

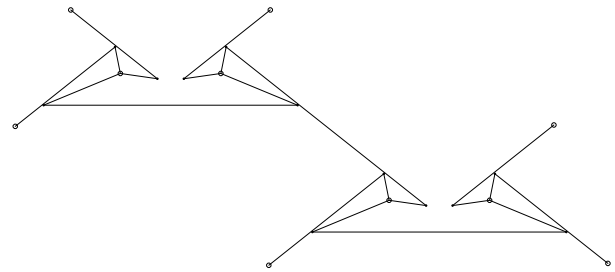


Figure 14: Family of plane geometric trees such that any augmentation leaves at least $\lfloor \frac{n-1}{10} \rfloor$ vertices without meeting their parity constraints.

5 Acknowledgment

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